APPENDIX C

COMPUTATION PROCEDURE FOR EXTREME VALUE (GUMBEL) DISTRIBUTION

(Reproduced from reference 48.)

5.2.6.1.2 *Computational methods*

It can be shown that most frequency functions applicable to hydrological analysis can take the form

$$X_{T_r} = \overline{X} + Ks_x \tag{5.1}$$

where $\overline{X}$ is the mean value, and s_x is the standard deviation of the variable being studied. Value X_{T_r} denotes the magnitude of the event reached or exceeded on an average once in T_r years. K is the frequency factor. If X is not normally distributed, K depends on frequency and skewness coefficient. A commonly used distribution of extreme values (annual series) is the double exponential distribution, which has been widely applied by Gumbel (see Bibliography), and often bears his name. In this method

$$K = \frac{Y_{T_r} - \overline{Y}_n}{s_n} \tag{5.2}$$

where $\overline{Y}_n$, the reduced mean, and s_n, the reduced standard deviation, are functions only of sample size; and Y_{T_r}, the reduced variate, is related to return period by

$$Y_{T_r} = -\left(0.83405 + 2.30259 \log \log \frac{T_r}{T_r - 1}\right) \qquad (5.3)$$

Table 5.3 gives values of K computed by means of Eq. (5.2) using Gumbel's values for $\overline{Y}_n$, s_n, and Y_{T_r}.

There are two basic methods for fitting data to the extreme value distribution. One consists in computation of X_{T_r} by means of Eq. (5.1), after a previous computation of the values of $\overline{X}$ and s_x (Table 5.4). The other consists in plotting data on suitable graph paper, known as extreme probability paper, and drawing a line by inspection.

TABLE 5.3

Values of K *based on Eq.* (5.2)

			Return period (years)			
n	2	5	10	25	50	100
10	—0.1355	1.0580	1.8483	2.8467	3.5874	4.3227
11	—0.1376	1.0338	1.8094	2.7894	3.5163	4.2379
12	—0.1393	1.0134	1.7766	2.7409	3.4563	4.1664
13	—0.1408	.9958	1.7484	2.6993	3.4048	4.1050
14	—0.1422	.9806	1.7240	2.6632	3.3600	4.0517
15	—0.1434	.9672	1.7025	2.6316	3.3208	4.0049
16	—0.1444	.9553	1.6835	2.6035	3.2860	3.9635
17	—0.1454	.9447	1.6665	2.5784	3.2549	3.9265
18	—0.1463	.9352	1.6512	2.5559	3.2270	3.8932
19	—0.1470	.9265	1.6373	2.5354	3.2017	3.8631
20	—0.1478	.9187	1.6247	2.5169	3.1787	3.8356
21	—0.1484	.9115	1.6132	2.4999	3.1576	3.8106
22	—0.1490	.9049	1.6026	2.4843	3.1383	3.7875
23	—0.1496	.8988	1.5929	2.4699	3.1205	3.7663
24	—0.1501	.8931	1.5838	2.4565	3.1040	3.7466
25	—0.1506	.8879	1.5754	2.4442	3.0886	3.7283
26	—0.1510	.8830	1.5676	2.4326	3.0743	3.7113
27	—0.1515	.8784	1.5603	2.4219	3.0610	3.6954
28	—0.1518	.8742	1.5535	2.4118	3.0485	3.6805
29	—0.1522	.8701	1.5470	2.4023	3.0368	3.6665

(continued)

TABLE 5.3 (continued)

Return period (years)

n	2	5	10	25	50	100
30	—0.1526	.8664	1.5410	2.3934	3.0257	3.6534
31	—0.1529	.8628	1.5353	2.3850	3.0153	3.6410
32	—0.1532	.8594	1.5299	2.3770	3.0054	3.6292
33	—0.1535	.8562	1.5248	2.3695	2.9961	3.6181
34	—0.1538	.8532	1.5199	2.3623	2.9873	3.6076
35	—0.1540	.8504	1.5153	2.3556	2.9789	3.5976
36	—0.1543	.8476	1.5110	2.3491	2.9709	3.5881
37	—0.1545	.8450	1.5068	2.3430	2.9633	3.5790
38	—0.1548	.8425	1.5028	2.3371	2.9561	3.5704
39	—0.1550	.8402	1.4990	2.3315	2.9491	3.5622
40	—0.1552	.8379	1.4954	2.3262	2.9425	3.5543
41	—0.1554	.8357	1.4920	2.3211	2.9362	3.5467
42	—0.1556	.8337	1.4886	2.3162	2.9301	3.5395
43	—0.1557	.8317	1.4854	2.3115	2.9243	3.5325
44	—0.1559	.8298	1.4824	2.3069	2.9187	3.5259
45	—0.1561	.8279	1.4794	2.3026	2.9133	3.5194
46	—0.1562	.8262	1.4766	2.2984	2.9081	3.5133
47	—0.1564	.8245	1.4739	2.2944	2.9031	3.5073
48	—0.1566	.8228	1.4712	2.2905	2.8983	3.5016
49	—0.1567	.8212	1.4687	2.2868	2.8937	3.4961
50	—0.1568	.8197	1.4663	2.2832	2.8892	3.4908
51	—0.1570	.8182	1.4639	2.2797	2.8849	3.4856
52	—0.1571	.8168	1.4616	2.2763	2.8807	3.4807
53	—0.1572	.8154	1.4594	2.2731	2.8767	3.4759
54	—0.1573	.8141	1.4573	2.2699	2.8728	3.4712
55	—0.1575	.8128	1.4552	2.2669	2.8690	3.4667
56	—0.1576	.8116	1.4532	2.2639	2.8653	3.4623
57	—0.1577	.8103	1.4512	2.2610	2.8618	3.4581
58	—0.1578	.8092	1.4494	2.2583	2.8583	3.4540
59	—0.1579	.8080	1.4475	2.2556	2.8550	3.4500
60	—0.1580	.8069	1.4458	2.2529	2.8518	3.4461
61	—0.1581	.8058	1.4440	2.2504	2.8486	3.4424
62	—0.1582	.8048	1.4424	2.2479	2.8455	3.4387
63	—0.1583	.8038	1.4407	2.2455	2.8426	3.4352
64	—0.1583	.8028	1.4391	2.2432	2.8397	3.4317

TABLE 5.3 (continued)

Return period (years)

n	2	5	10	25	50	100
65	—0.1584	.8018	1.4376	2.2409	2.8368	3.4284
66	—0.1585	.8009	1.4361	2.2387	2.8341	3.4251
67	—0.1586	.8000	1.4346	2.2365	2.8314	3.4219
68	—0.1587	.7991	1.4332	2.2344	2.8288	3.4188
69	—0.1587	.7982	1.4318	2.2324	2.8263	3.4158
70	—0.1588	.7974	1.4305	2.2304	2.8238	3.4128
71	—0.1589	.7965	1.4291	2.2284	2.8214	3.4099
72	—0.1590	.7957	1.4278	2.2265	2.8190	3.4071
73	—0.1590	.7950	1 4266	2.2246	2.8167	3.4044
74	—0.1591	.7942	1.4254	2.2228	2.8144	3.4017
75	—0.1592	.7934	1.4242	2.2211	2.8122	3.3991
76	—0.1592	.7927	1.4230	2.2193	2.8101	3.3965
77	—0.1593	.7920	1.4218	2.2176	2.8080	3.3940
78	—0.1593	.7913	1.4207	2.2160	2.8059	3.3916
79	—0.1594	.7906	1.4196	2.2143	2.8039	3.3892
80	—0.1595	.7899	1.4185	2.2128	2.8020	3.3868
81	—0.1595	.7893	1.4175	2.2112	2.8000	3.3845
82	—0.1596	.7886	1.4165	2.2097	2.7982	3.3823
83	—0.1596	.7880	1.4154	2.2082	2.7963	3.3801
84	—0.1597	.7874	1.4145	2.2067	2.7945	3.3779
85	—0.1597	.7868	1.4135	2.2053	2.7927	3.3758
86	—0.1598	.7862	1.4125	2.2039	2.7910	3.3738
87	—0.1598	.7856	1.4116	2.2026	2.7893	3.3717
88	—0.1599	.7851	1.4107	2.2012	2.7877	3.3698
89	—0.1599	.7845	1.4098	2.1999	2.7860	3.3678
90	—0.1600	.7840	1.4089	2.1986	2.7844	3.3659
91	—0.1600	.7834	1.4081	2.1973	2.7828	3.3640
92	—0.1601	.7829	1.4072	2.1961	2.7813	3.3622
93	—0.1601	.7824	1.4064	2.1949	2.7798	3.3604
94	—0.1602	.7819	1.4056	2.1937	2.7783	3.3586
95	—0.1602	.7814	1.4048	2.1925	2.7769	3.3569
96	—0.1602	.7809	1.4040	2.1913	2.7754	3.3552
97	—0.1603	.7804	1.4033	2.1902	2.7740	3.3535
98	—0.1603	.7800	1.4025	2.1891	2.7726	3.3519
99	—0.1604	.7795	1.4018	2.1880	2.7713	3.3503
100	—0.1604	.7791	1.4010	2.1869	2.7700	3.3487

Extreme probability paper has a linear ordinate for the variable being studied, and the abscissa is a linear scale of the reduced variate [Eq. (5.3)]. For convenience in plotting, the warped scale of T_r is also shown along the top of Fig. 5.9. Plotting positions are commonly determined by the formulae [16]:

$$T_r = \frac{n+1}{m} \tag{5.4}$$

or

$$T_r = \frac{n+0.4}{m-0.3} \tag{5.5}$$

where n is the number of years of record (the number of items in the annual series) and m is the rank of the item on the series, m being 1 for the largest.

To illustrate the steps in numerical computation of the rainfall value for a given return period, hypothetical values of a series of annual rainfall maxima are given in the upper part of Table 5.4. Computations are illustrated in the lower part of the table for T_r of 10. Rainfall depths for return periods other than 10 years can be computed in a similar manner.

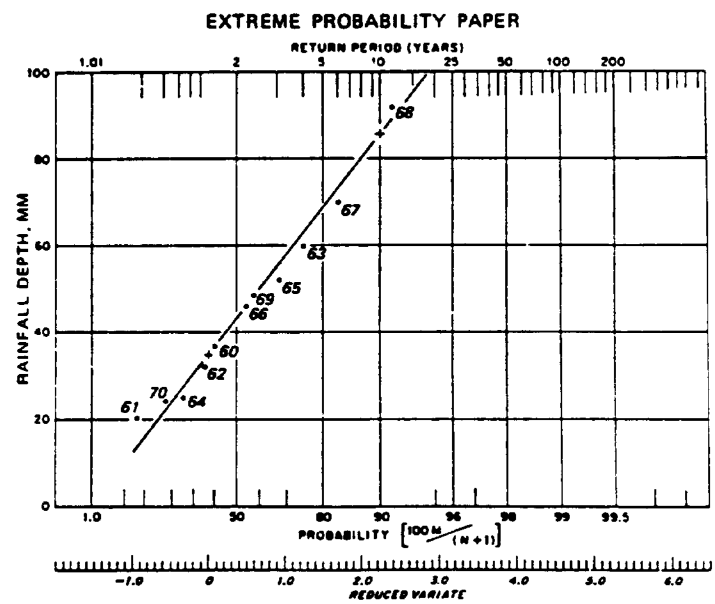

Figure 5.9 - Example of extreme probability plot using data of Table 5.4.

TABLE 5.4
Computation of extreme values

Year	P	m	$\frac{n+1}{m}$	$P-\overline{P}$	$(P-\overline{P})^2$	P^2
1960	37	7	1.72	−9	81	1,369
1961	20	11	1.09	−26	676	400
1962	32	8	1.5	−14	196	1,024
1963	60	3	4.0	+14	196	3,600
1964	25	9	1.33	−21	441	625
1965	52	4	3.0	+6	36	2,704
1966	46	6	2.0	0	0	2,116
1967	70	2	6.0	+24	576	4,900
1968	92	1	12.0	+46	2,116	8,464
1969	48	5	2.4	+2	4	2,304
1970	24	10	1.2	−22	484	576
Total	506				4,806	28,082

$$\overline{P} = \sum^{n} P/n = \frac{506}{11} = 46.0$$

s_x by square of deviations: $\sqrt{\frac{\sum^{n}(P)^2 - \overline{P}\sum^{n}P}{n-1}} = \sqrt{\frac{4806}{10}} = 21.92$

s_x by short cut: $\sqrt{\frac{\sum^{n}(P)^2 - \overline{P}\sum^{n}P}{n-1}} = \sqrt{\frac{4806}{10}} = 21.92$

For $T_r = 2$ and $n = 11$, $K = -0.1376$ (from Table 5.3).

Substituting into Eq. (5.1):

$$P_2 = 46.0 - 0.1376 \times 21.92 = 43.0$$

Similarly, for $T_r = 10$, $K = 1.8094$, and

$$P_{10} = 46.0 + 1.8094 \times 21.92 = 85.7$$

In Fig. 5.9 the two + 's show the above values for P_2 and P_{10} and define the line shown.

To illustrate the graphical method of fitting data to the extreme-value distribution, reference is again made to Table 5.4 and Fig. 5.9. In the table, values of the plotting position are given, and in Fig. 5.9 the plotted points are given, with rainfall values for each plotting position. The curve shown could have been drawn by fitting the plotted points by inspection.

In this example, for convenience, a record of only 11 years is used. Such a record gives a fairly stable value for return periods of as much as five years, but for longer return periods the short record has a large sampling error and the computations should not be taken as precise estimates.

For some types of data, instead of using the extreme-value distribution, a better fit of the data, or a closer approach to linearity, may be obtained from one of several other types of distribution, such as the normal or lognormal Pearson Type III distribution. A commonly used distribution is one in which the magnitude scale is logarithmic and the probability or return-period scale is the normal distribution. This distribution and the plotting paper used with it are widely known as log-normal. For discussion of additional distributions and of additional methods for fitting distributions, reference may be made to textbooks, and periodical statistical literature [17, 18, 63–66].

An advantage of fitting data to a distribution is achievement of objectivity. This advantage has the corollary of standard treatment of data, so that a decision is based on differences in data rather than differences in subjective interpretation of data. A third advantage of linearity in plotting points, and of close fit to a particular distribution, is the facility for extrapolating beyond the range of the data. However, it should be remembered that extrapolation involves considerable sampling error.

For evaluation of the accuracy of the computed values X_{T_r} it would be desirable to compute the confidence interval with limits:

$$X_{T_r} - t(\alpha)\, s_e; \qquad X_{T_r} + t(\alpha)\, s_e$$

within which, with given confidence levels, one may expect to find the true precipitation value X_{T_r}. Values of $t(\alpha)$ for selected confidence levels are as follows:

$\alpha = 95\,\%$	$t(\alpha) = 1.960$
$\alpha = 90\,\%$	$t(\alpha) = 1.645$
$\alpha = 80\,\%$	$t(\alpha) = 1.282$
$\alpha = 68\,\%$	$t(\alpha) = 1.000$

In most cases, values of s_e, the standard error of estimate, can be computed by means of the formula

$$s_e = \beta_{T_r} \cdot \frac{s_x}{\sqrt{n}} \tag{5.6}$$

In particular, for the Gumbel distribution the following relation [19] exists:

$$\beta_{T_r} = \sqrt{1 + 1.14K + 1.10K^2} \tag{5.7}$$

where K is the numerical value defined by Eq. (5.2) and readily obtainable from Table 5.3. However, for convenience in the use of Eq. (5.6) values of $\beta_{T_r}/\sqrt{n}$ can be obtained directly from Table 5.5. Thus, in determining the 80 per cent confidence interval for $P_{10} = 85.7$ in Table 5.4, for example,

$$t(\alpha)s_e = 1.282 \times 0.7783 \times 21.92 = 21.9$$

The lower and upper limits of the confidence interval are therefore 85.7 − 21.9 and 85.7 + 21.9, respectively, which means that there is an 80 per cent probability that the true value of P_{10} lies between 63.8 and 107.6. Similarly, the lower and upper limits of the same confidence interval for $P_2 = 43.0$ are 35.1 and 50.9, respectively, the value of $\beta_{T_r}/\sqrt{n}$ being 0.2803 (Table 5.5).

TABLE 5.5

Values of $\beta_{T_r}/\sqrt{n}$ for use in Eq. (5.6)

	Return period (years)					
n	2	5	10	25	50	100
10	.2942	.5863	.8285	1.1472	1.3873	1.6273
11	.2803	.5522	.7783	1.0761	1.3007	1.5252
12	.2681	.5232	.7358	1.0161	1.2275	1.4389
13	.2574	.4982	.6992	.9645	1.1646	1.3648
14	.2479	.4763	.6673	.9196	1.1100	1.3005
15	.2393	.4569	.6392	.8801	1.0620	1.2439
16	.2316	.4397	.6142	.8450	1.0193	1.1937
17	.2246	.4242	.5918	.8136	.9811	1.1488
18	.2182	.4102	.5716	.7853	.9467	1.1083
19	.2123	.3974	.5532	.7596	.9155	1.0716
20	.2068	.3857	.5365	.7361	.8871	1.0382
21	.2018	.3750	.5211	.7146	.8610	1.0075
22	.1971	.3651	.5069	.6948	.8370	.9793
23	.1927	.3559	.4937	.6765	.8148	.9532
24	.1886	.3473	.4815	.6595	.7942	.9290
25	.1847	.3394	.4702	.6437	.7750	.9064
26	.1811	.3319	.4595	.6289	.7571	.8854
27	.1777	.3249	.4496	.6150	.7403	.8657
28	.1745	.3183	.4402	.6020	.7245	.8472
29	.1714	.3121	.4314	.5898	.7097	.8298
30	.1685	.3062	.4230	.5782	.6957	.8134
31	.1657	.3007	.4152	.5673	.6825	.7978
32	.1631	.2954	.4077	.5569	.6699	.7831
33	.1606	.2904	.4006	.5471	.6581	.7692
34	.1582	.2856	.3938	.5377	.6468	.7559
35	.1559	.2811	.3874	.5289	.6360	.7433
36	.1537	.2767	.3813	.5204	.6257	.7313
37	.1516	.2726	.3754	.5123	.6159	.7198
38	.1496	.2686	.3698	.5045	.6066	.7088
39	.1476	.2648	.3645	.4971	.5976	.6983

(continued)

TABLE 5.5 (continued)

Return period (years)

n	2	5	10	25	50	100
40	.1457	.2611	.3593	.4900	.5890	.6883
41	.1439	.2576	.3544	.4832	.5808	.6786
42	.1422	.2542	.3496	.4766	.5729	.6693
43	.1405	.2510	.3451	.4703	.5653	.6604
44	.1389	.2479	.3407	.4643	.5580	.6518
45	.1373	.2449	.3365	.4584	.5509	.6436
46	.1358	.2419	.3324	.4528	.5441	.6356
47	.1344	.2391	.3284	.4474	.5375	.6279
48	.1330	.2364	.3246	.4431	.5312	.6205
49	.1316	.2338	.3209	.4370	.5251	.6133
50	.1303	.2312	.3174	.4321	.5192	.6064
51	.1290	.2288	.3139	.4274	.5134	.5996
52	.1277	.2264	.3106	.4228	.5079	.5931
53	.1265	.2241	.3073	.4183	.5025	.5868
54	.1253	.2218	.3042	.4140	.4973	.5807
55	.1242	.2196	.3012	.4098	.4922	.5748
56	.1230	.2175	.2982	.4057	.4873	.5690
57	.1219	.2155	.2953	.4018	.4825	.5635
58	.1209	.2135	.2925	.3979	.4779	.5580
59	.1198	.2115	.2898	.3942	.4734	.5527
60	.1188	.2096	.2872	.3905	.4690	.5476
61	.1179	.2078	.2846	.3870	.4647	.5426
62	.1169	.2060	.2821	.3836	.4606	.5377
63	.1160	.2042	.2796	.3802	.4565	.5330
64	.1150	.2025	.2772	.3769	.4525	.5284
65	.1142	.2008	.2749	.3737	.4487	.5239
66	.1133	.1992	.2726	.3706	.4449	.5195
67	.1124	.1976	.2704	.3676	.4413	.5152
68	.1116	.1960	.2683	.3646	.4377	.5110
69	.1108	.1945	.2661	.3617	.4342	.5069
70	.1100	.1930	.2641	.3589	.4208	.5029
71	.1092	.1916	.2621	.3561	.4274	.4990
72	.1084	.1902	.2601	.3534	.4242	.4952
73	.1077	.1888	.2582	.3507	.4210	.4914
74	.1070	.1874	.2563	.3481	.4179	.4878

TABLE 5.5 (continued)

Return period (years)

n	2	5	10	25	50	100
75	.1062	.1861	.2544	.3456	.4148	.4842
76	.1055	.1848	.2526	.3431	.4118	.4807
77	.1048	.1835	.2508	.3407	.4089	.4773
78	.1042	.1823	.2491	.3383	.4060	.4739
79	.1035	.1810	.2474	.3360	.4032	.4706
80	.1028	.1798	.2457	.3337	.4005	.4674
81	.1022	.1786	.2441	.3315	.3978	.4643
82	.1016	.1775	.2425	.3293	.3951	.4612
83	.1010	.1764	.2409	.3271	.3925	.4581
84	.1004	.1752	.2394	.3250	.3900	.4552
85	.0998	.1742	.2379	.3229	.3875	.4522
86	.0992	.1731	.2364	.3209	.3850	.4494
87	.0986	.1720	.2349	.3189	.3826	.4466
88	.0980	.1710	.2335	.3169	.3803	.4438
89	.0975	.1700	.2321	.3150	.3780	.4411
90	.0969	.1690	.2307	.3131	.3757	.4384
91	.0964	.1680	.2293	.3113	.3734	.4358
92	.0959	.1670	.2280	.3094	.3712	.4332
93	.0954	.1661	.2267	.3076	.3691	.4307
94	.0948	.1652	.2254	.3059	.3670	.4282
95	.0943	.1642	.2241	.3041	.3649	.4258
96	.0939	.1633	.2229	.3024	.3628	.4234
97	.0934	.1624	.2217	.3007	.3608	.4210
98	.0929	.1616	.2204	.2991	.3588	.4187
99	.0924	.1607	.2193	.2975	.3569	.4164
100	.0919	.1599	.2181	.2959	.3549	.4142

An example of the magnitude of error in extrapolation beyond the range of the data may be found in the record of maximum annual 24-hour rainfall at Hartford, Connecticut, U.S.A. Based on the 50 years of record through 1954, the 100-year value was found to be 155 mm. The maximum event during this period was 170 mm. In 1955 a hurricane produced 307 mm in 24 hours. The computation of 100-year 24-hour rainfall based on the 51 years of record through 1955 resulted in a new estimate of 218 mm, a 40 per cent increase. Even the 10-year value was increased substantially by this one event.

It may happen, however, that during a definite period of T_r years, precipitation of the magnitude $P \geqq P_{T_r}$ does not occur at all, or that it occurs several times. The probability that, during a given period of t years, a respective phenomenon will occur n times, is equal to

$$Pr_{n/t} = \left(\frac{t!}{n!\,(t-n)!}\right) p^n (1-p)^{t-n} \tag{5.8}$$

where $p = 1/T_r$. Assuming, for example, that $t = T_r = 100$ years, then the probabilities for various values of n are:

n	0	1	2	3	4	5
$Pr_{n/100}$	0.366	0.370	0.185	0.061	0.015	0.003

The overall probability of P_{T_r} or greater event occurring in t years is discussed in Sec. A.5.7.3.